{Sigma/Summation notation}

- $\sum_{i=1}^{4} A_i = A_1 + A_2 + A_3 + A_4$

  $\sum_{i=1}^{t} A_i = A_1 + A_2 + \dots + A_t$

- The i is just a placeholder
  - $\Rightarrow$ we could use any other letter
  - $\Rightarrow \sum_{i=1}^{t} A_i = \sum_{k=1}^{t} A_k$

- we can always "shift" indices

  $\sum_{i=1}^{4} A_i = A_1 + A_2 + A_3 + A_4$ $\qquad$ $\sum_{i=0}^{3} A_{i+1} = A_{0+1} + A_{1+1} + A_{2+1} + A_{3+1}$
  $= A_1 + A_2 + A_3 + A_4$

  $\Rightarrow \sum_{i=1}^{4} A_i = \sum_{i=0}^{3} A_{i+1}$

- Similarly;
  - $\sum_{i=1}^{t} A_i = \sum_{k=2}^{t+1} A_{k-1}$ (letting $k = i+1$)
  - $\sum_{i=1}^{t} A_i = \sum_{l=0}^{t-1} A_{l+1}$ (letting $l = i-1$)

    $= \sum_{\substack{s=0 \\ s \text{ even}}}^{2(t-1)} A_{\frac{s}{2}+1}$ (letting $\frac{s}{2} = l$)

- Adding sigma notation;
  - $\sum_{k=1}^{t} A_k + \sum_{k=1}^{t} B_k = \sum_{k=1}^{t} (A_k + B_k)$
  - $\sum_{k=1}^{t} A_k + \sum_{l=1}^{t} B_l = \sum_{k=1}^{t} A_k + \sum_{k=1}^{t} B_k = \sum_{k=1}^{t} (A_k + B_k)$

    replace l with k in B sum

    add as normal
  - $\sum_{k=1}^{t} A_k + \sum_{l=0}^{t-1} B_l = \sum_{k=1}^{t} A_k + \sum_{k=1}^{t} B_{k-1} = \sum_{k=1}^{t} (A_k + B_{k-1})$

    replace l with $k-1$ in B sum

    add as normal

- $\sum_{k=1}^{t} A_k + \sum_{l=0}^{t} B_l = \sum_{k=1}^{t} A_k + \sum_{l=1}^{t} B_l + B_0$ pull out B_0 term from sum

$= \sum_{k=1}^{t} A_k + \sum_{k=1}^{t} B_k + B_0$ replace l with k in B sum

$= \sum_{k=1}^{t} (A_k + B_k) + B_0$ add as normal

- $\sum_{k=1}^{t} A_k + \sum_{l=0}^{t} B_l = \sum_{k=1}^{t} A_k + \sum_{k=1}^{t+1} B_{k-1}$ replace l with $k-1$

$= \sum_{k=1}^{t} A_k + \sum_{k=1}^{t} B_{k-1} + B_{(t+1)-1}$ pull out $B_{(t+1)-1}$ term

$= \sum_{k=1}^{t} (A_k + B_{k-1}) + B_t$ add as normal

$$\Rightarrow \underbrace{\sum_{k=1}^{t} (A_k + B_k) + B_0 = \sum_{k=1}^{t} A_k + \sum_{l=0}^{t} B_l = \sum_{k=1}^{t} (A_k + B_{k-1}) + B_t}$$

There are lots of ways to express the same thing

- Another way to view the eqn above;

$B_0 + (A_1 + B_1) + \dots + (A_t + B_t) = (A_1 + \dots + A_t) + (B_0 + B_1 + \dots + B_t)$

$= (A_1 + B_0) + (A_2 + B_1) + \dots + (A_t + B_{t-1}) + B_t$

- uses for sigma notation;

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} a_{11}v_1 + a_{12}v_2 + a_{13}v_3 \\ a_{21}v_1 + a_{22}v_2 + a_{23}v_3 \\ a_{31}v_1 + a_{32}v_2 + a_{33}v_3 \end{pmatrix}$$

matrix-vector multiplication

$$= \begin{pmatrix} \sum_{j=1}^{3} a_{1j} v_j \\ \sum_{j=1}^{3} a_{2j} v_j \\ \sum_{j=1}^{3} a_{3j} v_j \end{pmatrix}$$

gather together the sums into sigma notation

$e_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}$ $e_2 = \begin{pmatrix}0\\1\\0\end{pmatrix}$ $e_3 = \begin{pmatrix}0\\0\\1\end{pmatrix}$

$$\longrightarrow = \left(\sum_{j=1}^{3} a_{1j} V_j\right) e_1 + \left(\sum_{j=1}^{3} a_{2j} V_j\right) e_2 + \left(\sum_{j=1}^{3} a_{3j} V_j\right) e_3$$

$$= \sum_{i=1}^{3} \sum_{j=1}^{3} a_{ij} V_j e_i$$

So this gives us a compact way to show matrix-vector multiplication.

Can you do the same for matrix-matrix multiplication? (Let me know if you have any trouble)

- Binomial expansion:

$$(x+y)^2 = x^2 + 2xy + y^2$$

$$(x+y)^3 = \underline{1}x^3 + \underline{3}x^2y + \underline{3}xy^2 + \underline{1}y^3$$

The coefficents are represented in pascal's triangle

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & \underline{1} & & \underline{3} & & \underline{3} & & \underline{1} & \\ 1 & & 4 & & 6 & & 4 & & 1 \\ & & & & \vdots & & & & \end{array}$$

$$\binom{n}{0} \quad \binom{n}{1} \cdots \binom{n}{k} \cdots \binom{n}{n-1} \quad \binom{n}{n}$$

$$\binom{n}{0} = 1, \quad \binom{n}{1} = n, \quad \binom{n}{k} = \frac{n!}{k!(n-k)!}, \quad \binom{n}{n-1} = n, \quad \binom{n}{n} = 1$$

- $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$

- eg for $n=6$

$$(x+y)^6 = \sum_{k=0}^{6} \binom{6}{k} x^{6-k} y^k$$

Have a go at calculating this value

$$= \binom{6}{0} x^{6-0} y^0 + \binom{6}{1} x^{6-1} y^1 + \binom{6}{2} x^{6-2} y^2 + \binom{6}{3} x^{6-3} y^3$$
$$+ \binom{6}{4} x^{6-4} y^4 + \binom{6}{5} x^{6-5} y^5 + \binom{6}{6} x^{6-6} y^6$$

$$= x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4$$
$$+ 6xy^5 + y^6$$