Inclusion Exclusion Examples + proofs

Two Sets

- Theorem - $|A \cup B| = |A| + |B| - |A \cap B|$
- Example -
  - Anna, Ben, claire, Dan + Ellen study maths
  - Fred, Gemma, Dan + Ellen study physics
  - calculate the number of people that study maths or physics

  - Let M = maths students
    P = physics students

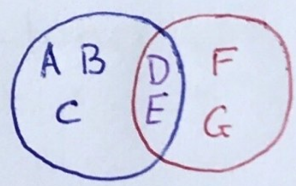
    $M = \{$Anna, Ben, claire, Dan, Ellen$\}$ ← notice Dan and Ellen are in both sets
    $P = \{$Fred, Gemma, Dan, Ellen$\}$ ←

    $M \cup P = \{$Anna, Ben, claire, Dan, Ellen, Fred, Gemma$\}$ But they each only apper once in the union

    $M \cap P = \{$Dan, Ellen$\}$

    $|M| = 5 \quad |P| = 4 \quad |M \cup P| = 7 \quad |M \cap P| = 2$

    $$\Rightarrow |M \cup P| = |M| + |P| - |M \cap P|$$
    $$7 = 5 + 4 - 2$$



Three sets

- Theorem - $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$
- Example -
  - Anna, Ben, claire, Dan, Ellen + Lily study maths
  - Fred, Gemma, Dan, Ellen + Ian study physics
  - Ian, Jim, Dan, kelly + Lily study chemistry
  - calculate the number of people that study maths or physics or chemistry

- Let M = maths students
  P = physics students
  C = chemistry students

$\Rightarrow M = \{\text{Anna, Ben, claire, Dan, Ellen, Lily}\}$ $\quad |M| = 6$

$P = \{\text{Fred, Gemma, Dan, Ellen, Ian}\}$ $\quad |P| = 5$

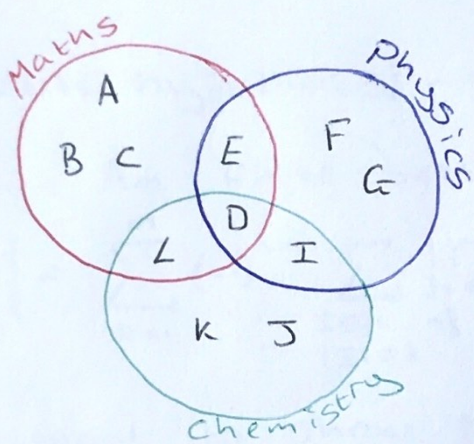
$C = \{\text{Ian, Jim, Dan, kelly, Lily}\}$ $\quad |C| = 5$

$M \cap P = \{\text{Ellen, Dan}\}$ $\quad |M \cap P| = 2$

$M \cap C = \{\text{Lily, Dan}\}$ $\quad |M \cap C| = 2$

$P \cap C = \{\text{Dan, Ian}\}$ $\quad |P \cap C| = 2$

$M \cap P \cap C = \{\text{Dan}\}$ $\quad |M \cap P \cap C| = 1$



$$|M \cup P \cup C| = |M| + |P| + |C| - |M \cap P| - |M \cap C| - |P \cap C| + |M \cap P \cap C|$$
$$11 \quad = 6 + 5 + 5 - 2 - 2 - 2 + 1$$

$\Rightarrow |M \cup P \cup C| = 11$

Check;

$M \cup P \cup C = \{\text{Anna, Ben, claire, Dan, Ellen, Fred, Gemma, Ian, Jim, kelly, Lily}\}$

$|M \cup P \cup C| = 1$:)

n sets

- Theorem - Let $A_1, A_2, \ldots, A_n$ be finite sets

$$|A_1 \cup \ldots \cup A_n| = \sum_{k=1}^{n} (-1)^{k-1} \sum_{\substack{I \subseteq \{1 \ldots n\} \\ |I| = k}} \left| \bigcap_{i \in I} A_i \right|$$

run over all subsets of $\{1 \ldots n\}$ of size k. There are $\binom{n}{k}$ of these

Proof This is a pretty hard proof so don't worry if you don't understand it first time

- $n=1$

$$\sum_{k=1}^{1} (-1)^{k-1} \sum_{\substack{I \subseteq \{1\} \\ |I|=1}} \left| \bigcap_{i \in I} A_i \right| = (-1)^{1-1} |A_1| = |A_1|$$

- $n=2$

see thm 3.1 in notes (let me know if you have any trouble with this)

- IH (inductive hypothesis) - suppose the theorem holds for $m \geq 2$

$\Rightarrow$ For $A_1 \ldots A_m$ finite sets

$$\left| \bigcup_{i \in \{1 \ldots m\}} A_i \right| = \sum_{k=1}^{m} (-1)^{k-1} \sum_{\substack{I \subseteq \{1 \ldots m\} \\ |I|=k}} \left| \bigcap_{i \in I} A_i \right| \quad \text{—— (1)}$$

- We now want to show that the theorem holds for $m+1$

consider $A_1, A_2, \ldots, A_m, A_{m+1}$ a collection of $m+1$ sets

$\Rightarrow (A_1 \cap A_{m+1}), (A_2 \cap A_{m+1}), \ldots, (A_m \cap A_{m+1})$ is a collection of m sets

$\Rightarrow$ we can apply the IH to these m sets

$$\Rightarrow \left| \bigcup_{i \in \{1 \ldots m\}} (A_i \cap A_{m+1}) \right| = \sum_{k=1}^{m} (-1)^{k-1} \sum_{\substack{I \subseteq \{1 \ldots m\} \\ |I|=k}} \left| \bigcap_{i \in I} (A_i \cap A_{m+1}) \right|$$

$$= \sum_{k=1}^{m} (-1)^{k-1} \sum_{\substack{I \subseteq \{1 \ldots m\} \\ |I|=k}} \left| \left(\bigcap_{i \in I} A_i \right) \cap A_{m+1} \right|$$

$$= \sum_{k=1}^{m} (-1)^{k-1} \sum_{\substack{I \subseteq \{1 \ldots m\} \\ |I|=k}} \left| \bigcap_{i \in I \cup \{m+1\}} A_i \right|$$

$$= \sum_{k=1}^{m} (-1)^{k-1} \sum_{\substack{J \subseteq \{1 \dots m, m+1\} \\ m+1 \in J \\ |J| = k+1}} \left| \bigcap_{j \in J} A_j \right|$$

Let $J = I \cup \{m+1\}$

$$= \sum_{l=2}^{m+1} (-1)^{l-2} \sum_{\substack{J \subseteq \{1 \dots m+1\} \\ m+1 \in J \\ |J| = l}} \left| \bigcap_{j \in J} A_j \right|$$

let $l = k+1$

②

Consider

$$\left| \bigcup_{i \in \{1 \dots m+1\}} A_i \right| = \left| \underbrace{\left(\bigcup_{i \in \{1 \dots m\}} A_i \right)}_{\text{Set 1}} \cup \underbrace{A_{m+1}}_{\text{Set 2}} \right|$$

Apply I/E to the above (weve checked the result holds for two sets in T3.1)

$$\Rightarrow \left| \bigcup_{i \in \{1 \dots m+1\}} A_i \right| = \underbrace{\left| \bigcup_{i \in \{1 \dots m\}} A_i \right|}_{\text{we have an expression for this by ①}} + \left| A_{m+1} \right| - \underbrace{\left| \left(\bigcup_{i \in \{1 \dots m\}} A_i \right) \cap A_{m+1} \right|}_{\text{we have an expression for this by ②}}$$

$$= \sum_{k=1}^{m} (-1)^{k-1} \sum_{\substack{I \subseteq \{1 \dots m\} \\ |I| = k}} \left| \bigcap_{i \in I} A_i \right| + |A_{m+1}|$$

$$- \sum_{l=2}^{m+1} (-1)^{l-2} \sum_{\substack{J \subseteq \{1 \dots m+1\} \\ m+1 \in J \\ |J| = l}} \left| \bigcap_{j \in J} A_j \right|$$

These two expressions are the same. Can you check?

$$= \sum_{k=1}^{m} (-1)^{k-1} \sum_{\substack{I \subseteq \{1 \dots m\} \\ |I| = k}} \left| \bigcap_{i \in I} A_i \right| + (-1)^{1-1} \sum_{\substack{J \subseteq \{1 \dots m+1\} \\ m+1 \in J \\ |J| = 1}} \left| \bigcap_{j \in J} A_j \right|$$

$-(-1)^{l-2} = (-1)^{l-1}$

$$+ \sum_{l=2}^{m+1} (-1)^{l-1} \sum_{\substack{J \subseteq \{1 \dots m+1\} \\ m+1 \in J \\ |J| = l}} \left| \bigcap_{j \in J} A_j \right|$$

$$= \sum_{k=1}^{m} (-1)^{k-1} \sum_{\substack{I \subseteq \{1 \dots m\} \\ |I| = k}} \left| \bigcap_{i \in I} A_i \right| + \sum_{l=1}^{m+1} (-1)^{l-1} \sum_{\substack{J \subseteq \{1 \dots m+1\} \\ m+1 \in J \\ |J| = l}} \left| \bigcap_{j \in J} A_j \right|$$

$$= \sum_{k=1}^{m+1} (-1)^{k-1} \sum_{\substack{I \subseteq \{1 \dots m+1\} \\ |I| = k \\ m+1 \notin I}} \left| \bigcap_{i \in I} A_i \right| + \sum_{k=1}^{m+1} (-1)^{k-1} \sum_{\substack{I \subseteq \{1 \dots m+1\} \\ m+1 \in I \\ |I| = k}} \left| \bigcap_{j \in J} A_j \right|$$

First term -

- The collections of subsets of $\{1 \dots m\}$ is the same as the collection of subsets of $\{1 \dots m+1\}$ which do not contain $m+1$
- There are no sets of $\{1 \dots m+1\}$ which do not contain $m+1$ so we can change the summand from $k=1$ to m to $k=1$ to $m+1$ without changing the expression

Second term - let $I = J$ and $l = k$

$$= \sum_{k=1}^{m+1} (-1)^{k-1} \left[\underbrace{\sum_{\substack{I \subseteq \{1 \dots m+1\} \\ |I| = k \\ m+1 \notin I}} \left| \bigcap_{i \in I} A_i \right|}_{A} + \underbrace{\sum_{\substack{I \subseteq \{1 \dots m+1\} \\ |I| = k \\ m+1 \in I}} \left| \bigcap_{j \in J} A_j \right|}_{B} \right]$$

we've just added together the terms

$$= \sum_{k=1}^{m+1} (-1)^{k-1} \underbrace{\sum_{\substack{I \subseteq \{1 \dots m+1\} \\ |I| = k}} \left| \bigcap_{i \in I} A_i \right|}_{C}$$

a set either contains $m+1$ or does not. So we can add A and B together to get C

$$\Rightarrow \left| \bigcup_{i \in \{1 \dots m+1\}} A_i \right| = \sum_{k=1}^{m+1} (-1)^{k-1} \sum_{\substack{I \subseteq \{1 \dots m+1\} \\ |I| = k}} \left| \bigcap_{i \in I} A_i \right| \qquad \square$$

(Thank God that's over!)