Recursion + Fibonacci Numbers

Guesstimating F(n)

example 1

$F(0) = 1 \qquad F(n) = 5F(n-1)$

Let's try some small examples;

$$\left.\begin{aligned} F(0) &= 1 \\ F(1) &= 5 \cdot 1 = 5 \\ F(2) &= 5 \cdot 5 = 5^2 \\ F(3) &= 5 \cdot 5^2 = 5^3 \\ F(4) &= 5 \cdot 5^3 = 5^4 \end{aligned}\right\} \text{Try + spot a pattern}$$

Guess – $F(n) = 5^n$

Prove by the Guess by induction

$F(0) = 1 = 5^0 \Rightarrow$ true for base case

Assume $F(n) = 5^n$

$$\begin{aligned} F(n+1) &= 5 \cdot F(n) \\ &= 5 \cdot 5^n \\ &= 5^{n+1} \end{aligned}$$

$\Rightarrow$ Guess holds by induction

Example 2

$F(0) = 2 \qquad F(n) = \frac{F(n-1)}{n}$

Small examples;

$F(0) = 2$

$F(1) = \frac{2}{1}$

$F(2) = \frac{2/1}{2} = 1$

Patten spotting

$2 = \frac{2}{0!}$

$2 = 2/1!$

$1 = 2/2!$

$f(3) = \frac{1}{3}$ $\qquad\qquad$ $1/3 = 2/3!$

$f(4) = \frac{1/3}{4} = \frac{1}{4 \cdot 3}$ $\qquad\qquad$ $\frac{1}{4 \cdot 3} = 2/4!$

$f(5) = \frac{1/4\cdot 3}{5} = \frac{1}{5 \cdot 4 \cdot 3}$ $\qquad\qquad$ $\frac{1}{5 \cdot 4 \cdot 3} = 2/5!$

$\Rightarrow$ Guess $f(n) = 2/n!$

prove by induc

$f(0) = 2 = \frac{2}{1} = \frac{2}{0!}$ $\Rightarrow$ BC holds

assume $f(n) = \frac{2}{n!}$

$$f(n+1) = \frac{f(n)}{n+1}$$
$$= \frac{2/n!}{n+1}$$
$$= \frac{2}{(n+1)\cdot n!}$$
$$= \frac{2}{(n+1)!}$$

$\Rightarrow$ holds by induction

<u>Example 3</u>

Let's try a harder example

$f(0) = 1 \qquad f(n) = 5 + 2f(n-1)$

$f(0) = 1$

$f(1) = 5 + 2(1)$

$f(2) = 5 + 2(5 + 2(1)) = 5(1+2) + 2^2$

$$f(3) = 5 + 2\left(5(1+2) + 2^2\right) = 5 + 5(2+2^2) + 2^3$$
$$= 5(1+2+2^2) + 2^3$$

$$f(4) = 5 + 2\left(5(1+2+2^2) + 2^3\right)$$
$$= 5\left(1+2+2^2+2^3\right) + 2^4$$

Guess $f(n) = 5\left(\underbrace{1+2+2^2+\dots+2^{n-1}}_{= 2^n - 1}\right) + 2^n$

See the end of the handout

$$= 5(2^n - 1) + 2^n$$

Proof

BC $f(0) = 1$

$5(2^0 - 1) + 2^0 = 5(1-1) + 1 = 1$ ✓

assume $f(n) = 5(2^n - 1) + 2^n$

$$f(n+1) = 5 + 2f(n)$$
$$= 5 + 2\left[5(2^n - 1) + 2^n\right]$$
$$= 5 + 5(2^{n+1} - 2) + 2^{n+1}$$
$$= 5(2^{n+1} - 2 + 1) + 2^{n+1}$$
$$= 5(2^{n+1} - 1) + 2^{n+1}$$

Top Tip – In the small examples gather together the "like" terms (eg powers of 2 and multiples of 5) but don't fully expand / type into calc. This makes pattern spotting easier!

December 2014

Q4)

4. An infinite sequence a_n is defined by $a_0 = 1/2$, and $a_n = 2^{n-1} + 3a_{n-1}$ for $n \geq 1$.

(a) Calculate a_1, a_2, a_3. [2]

(b) Calculate the generating function $f(x)$ of the sequence $b_n = 2^n$, for $n \geq 0$. [2]

(c) Let $g(x)$ be the generating function of the sequence a_n defined above. Show that

$$g(x) = \frac{1}{2(1-2x)(1-3x)}.$$

[3]

(d) Express $g(x)$ as a sum of two fractions with linear polynomials in their denominators. [2]

(e) Hence find a closed form expression for a_n, for all n. [2]

(f) Calculate a_{10}. [1]

a) $a_0 = 1/2$

$a_1 = 2^{1-1} + 3(1/2) = 1 + 3/2 = 5/2$

$a_2 = 2^{2-1} + 3(5/2) = 2 + \frac{15}{2} = 19/2$

$a_3 = 2^{3-1} + 3\left(\frac{19}{2}\right) = 4 + \frac{56}{2} = \frac{64}{2} = 32$

b) $b_n = 2^n \quad n \geq 0$

$$F(x) = \sum_{i=0}^{\infty} b_i x^i$$

$$= \sum_{i=0}^{\infty} 2^i x^i$$

$$= \sum_{i=0}^{\infty} (2x)^i$$

$$= \frac{1}{1-2x} \quad \text{(by thm 3.3)}$$

c) $$g(x) = \sum_{i=0}^{\infty} a_i x^i$$

$$= a_0 x^0 + \sum_{i=1}^{\infty} a_i x^i$$

$\boxed{\text{we want to use the recursive formula which uses } a_{n-1} \Rightarrow \text{pull } 1^{st} \text{ term out}}$

$$= \tfrac{1}{2}\cdot 1 + \sum_{i=1}^{\infty} (2^{i-1} + 3a_{i-1}) x^i$$

if the rec form uses a_{n-1}, a_{n-2} $\Rightarrow$ pull out 1^{st} + 2^{nd} term for ex see next q

Trying to make the 1^{st} sum look like part (b) and the 2^{nd} sum look like $g(x)$

$$= \tfrac{1}{2} + \sum_{i=1}^{\infty} 2^{i-1} x^i + 3\sum_{i=1}^{\infty} a_{i-1} x^i$$

Let $j = i-1$

$$= \tfrac{1}{2} + \sum_{j=0}^{\infty} 2^j x^{j+1} + 3\sum_{j=0}^{\infty} a_j x^{j+1}$$

$$= \tfrac{1}{2} + x\sum_{j=0}^{\infty} (2x)^j + 3x\sum_{j=0}^{\infty} a_j x^j$$

$$= \tfrac{1}{2} + \frac{x}{1-2x} + 3x\, g(x)$$

$$\Rightarrow \quad (1-3x)g(x) = \frac{1}{2} + \frac{x}{1-2x}$$

$$= \frac{1-2x+2x}{2(1-2x)}$$

$$= \frac{1}{2(1-2x)}$$

$$\Rightarrow \quad g(x) = \frac{1}{2(1-2x)(1-3x)}$$

d) use partial fractions

$$\frac{1}{2(1-2x)(1-3x)} = \frac{A}{1-2x} + \frac{B}{1-3x}$$

multiply by $(1-2x)(1-3x)$

$$\Leftrightarrow \quad \frac{1}{2} = A(1-3x) + B(1-2x)$$

Let $x = \frac{1}{2}$ $\Rightarrow$ $\frac{1}{2} = A\left(1 - 3\left(\frac{1}{2}\right)\right) + 0$

so that
$B(1-2x) = 0$

$$= -\frac{1}{2} A$$

$$\Rightarrow \quad A = -1$$

Let $x = \frac{1}{3}$ $\Rightarrow$ $\frac{1}{2} = B\left(1 - \frac{2}{3}\right)$

$$= \frac{1}{3} B$$

so that
$A(1-3x) = 0$ $\quad \Rightarrow \quad B = \frac{3}{2}$

$$\Rightarrow \quad g(x) = \frac{-1}{1-2x} + \frac{3}{2(1-3x)}$$

Hint - to check - type
$g(\pi) - \left[\frac{-1}{1-2\pi} + \frac{3}{2(1-3\pi)}\right]$ into
your calculator and check you get zero

e) recall $g(x) = \sum_{i=0}^{\infty} a_i x^i$

$$= a_0 x^0 + a_1 x^1 + a_2 x^2 + \dots$$

$$g(x) = \frac{-1}{1-2x} + \frac{3}{2(1-3x)}$$

$$= -1\left(\frac{1}{1-2x}\right) + \frac{3}{2}\left(\frac{1}{1-3x}\right)$$

$$= -1\left(1 + 2x + 2^2x^2 + 2^3x^3 + \dots\right)$$
$$+ \tfrac{3}{2}\left(1 + 3x + 3^2x^2 + 3^3x^3 + \dots\right)$$

$$= \left(-1 + \tfrac{3}{2}\right) + \left(-2 + \tfrac{3^2}{2}\right)x + \left(-2^2 + \tfrac{3^3}{2}\right)x^2$$
$$+ \left(-2^3 + \tfrac{3^4}{2}\right)x^3 + \dots$$

$$= a_0 + a_1x + a_2x^2 + a_3x^3$$

$\Rightarrow$ Guess

$$a_n = -2^n + \frac{3^{n+1}}{2}$$

base case $\quad a_0 = \tfrac{1}{2}$

$$-2^0 + \frac{3^1}{2} = -1 + \tfrac{3}{2} = \tfrac{1}{2} \quad \checkmark$$

assume $\quad a_n = -2^n + \dfrac{3^{n+1}}{2}$

$$a_{n+1} = 2^{(n+1)-1} + 3a_n$$
$$= 2^n + 3\left[-2^n + \frac{3^{n+1}}{2}\right] \quad \text{by IH}$$
$$= 2^n + 3(-2^n) + \frac{3^{(n+1)+1}}{2}$$
$$= 2^n(1-3) + \frac{3^{(n+1)+1}}{2}$$
$$= -2 \cdot 2^n + \frac{3^{(n+1)+1}}{2}$$
$$= -2^{n+1} + \frac{3^{(n+1)+1}}{2} \quad \Rightarrow \text{holds } \forall n$$

f) $a_{10} = -2^{10} + \frac{3^{11}}{2}$

August 2015

5. We can define the Fibonacci numbers by $f_0 = 0$, $f_1 = 1$, and $f_n = f_{n-1} + f_{n-2}$ for $n \geq 2$.

(a) State the definition of the *generating function* of a sequence a_n, for $n \geq 0$. **[1]**

(b) Write down the first four non-zero terms of the generating function $g(x)$ for the Fibonacci numbers. **[2]**

(c) Prove that the generating function $g(x)$ for the Fibonacci numbers satisfies

$$g(x) = \frac{x}{1 - x - x^2}.$$

[3]

(d) Show that the number of sequences of 0s and 1s of length n with the property that an odd number of consecutive 1s never occur is equal to the Fibonnaci number f_{n+1}. **[3]**

a) $g(x) = \sum_{i=0}^{\infty} a_i x^i$

b)
$$g(x) = f_0 x^0 + f_1 x^1 + f_2 x^2 + f_3 x^3 + f_4 x^4 + \dots$$
$$= 0 \cdot x^0 + 1x^1 + 1x^2 + 2x^3 + 3x^4 + \dots$$
$$= 0 + \underbrace{x + x^2 + 2x^3 + 3x^4}_{\text{First 4 non-zero terms}} + \dots$$

c)
$$g(x) = \sum_{i=0}^{\infty} f_i x^i$$

Pull out 1st two terms since f_n uses f_{n-2} in recursive def^n

$$= f_0 x^0 + f_1 x^1 + \sum_{i=2}^{\infty} f_i x^i$$
$$= 0 + x + \sum_{i=2}^{\infty} (f_{i-1} + f_{i-2}) x^i$$
$$= x + \sum_{i=2}^{\infty} f_{i-1} x^i + \sum_{i=2}^{\infty} f_{i-2} x^i$$

make a change of variable to turn f_{i-1} into f_j and f_{i-2} into f_k

$$= x + \sum_{j=1}^{\infty} f_j x^{j+1} + \sum_{k=0}^{\infty} f_k x^{k+2}$$

$\boxed{\begin{array}{l} \text{Let} \\ j = i-1 \\ k = i-2 \end{array}}$

Pull out copies of x to get $f_j x^j$ and $f_k x^k$

$$= x + \underbrace{0}_{= x \cdot f_0 x^0} + x \sum_{j=1}^{\infty} f_j x^j + x^2 \sum_{k=0}^{\infty} f_k x^k$$

because $f_0 = 0$

we can add in the $j=0$ term

$$= x + x \sum_{j=0}^{\infty} f_j x^j + x^2 \sum_{k=0}^{\infty} f_k x^k$$

$$= x + x g(x) + x^2 g(x)$$

$$\Rightarrow (1 - x - x^2) g(x) = x$$

$$\Rightarrow g(x) = \frac{x}{1 - x - x^2}$$

d) Let A_n be the set of words of length n with no odd number of consecutive 1s

$A_1 = \{0\}$

$A_2 = \{00, 11\}$

$A_3 = \{000, 110, 011\}$

$1 \notin A_1$ since 1 is an odd seq of consecutive 1s

Let's try and show $|A_n| = f_{n+1}$ by induction

$$\Rightarrow \left.\begin{array}{l} |A_1| = 1 = f_2 \\ |A_2| = 2 = f_3 \end{array}\right\} \longrightarrow \text{holds for base case}$$

$|A_3| = 3 = f_4$

Assume that

$|A_{n-1}| = f_n \qquad |A_{n-2}| = f_{n-1}$

$f_{n+1} = f_n + f_{n-1}$
$\Rightarrow$ need to assume 2 things in IH

consider A_n

Let $x_1 x_2 \cdots x_n \in A_n$

case 1 $-$ $x_n = 0$

$\Rightarrow x_1 x_2 \cdots x_{n-1} \in A_{n-1}$

case 2 $-$ $x_n = 1$

since $x_1 \cdots x_n \in A_n$

$\Rightarrow x_{n-1} = 1$ and $\exists i$ st

$$\Rightarrow \underbrace{x_{n-i}\, x_{n-i+1} \cdots x_{n-1}\, x_n}_{\text{contains only even sequece of 1s}}$$

$$\Rightarrow \underbrace{x_{n-i}\, x_{n-i+1} \cdots x_{n-1}}_{\text{contains odd seq of 1s}}$$

$$\Rightarrow \underbrace{x_{n-i}\, x_{n-i+1} \cdots x_{n-2}}_{\text{contains only even seq of 1s}}$$

$\Rightarrow x_1 x_2 \cdots x_{n-2} \in A_{n-2}$

$\Rightarrow \underline{x} \in A_n$

$$\Rightarrow \underline{x} = \begin{cases} \underline{y}0 & \underline{y} \in A_{n-1} \\ \underline{y}11 & \underline{y} \in A_{n-2} \end{cases}$$

$$\Rightarrow |A_n| = |A_{n-1}| + |A_{n-2}|$$
$$= f_n + f_{n-1}$$
$$= f_{n+1}$$

$\Rightarrow$ holds $\forall n$

Really what we're doing here is constructing a bijection between A_n and $A_{n-1} \cup A_{n-2}$.
Can you construct the bijection?

General idea of staircase

n stairs how many ways to get to the top of the staircase using 1-steps and 2-steps

(As was pointed out = number of ways to add 1 and 2 together to get n)

assume you choose to take 2-steps i times

$\Rightarrow$ $n-2i$ steps remaining

$\Rightarrow$ Take 2-steps i times
Take 1-steps $n-2i$ times

$\Rightarrow$ total number of "climbs"
(where a climb is taking 1-step or 2-step)
is $i+n-2i = n-i$

Just remember that

$$0 \le i \le \lfloor \tfrac{n}{2} \rfloor$$

you can do a 2-step zero times

the floor function $\lfloor \cdot \rfloor$ accounts for the possibility of an odd number of stairs

eg 9 stairs
you can take 2-steps atmost $4 = \lfloor \tfrac{9}{2} \rfloor$ times

Similar to choose a class of size r from m maths students + p physics students

Let i = number of phys in class
$\Rightarrow$ $n-i$ maths stu in class
where $0 \le i \le n$

$$2^n - 1 = 1 + 2 + 2^2 + \cdots + 2^{n-1}$$

Idea

$$2^n = 2 \cdot 2^{n-1}$$
$$= 2^{n-1} + 2^{n-1}$$
$$= 2 \cdot 2^{n-2} + 2^{n-1}$$
$$= 2^{n-2} + 2^{n-2} + 2^{n-1}$$
$$= 2 \cdot 2^{n-3} + 2^{n-2} + 2^{n-1}$$
$$= 2^{n-3} + 2^{n-3} + 2^{n-2} + 2^{n-1}$$
$$= \cdots$$
$$= 2 + 2 + 2^2 + 2^3 + \cdots + 2^{n-1}$$

$$\Rightarrow 2^n - 1 = 1 + 2 + 2^2 + 2^3 + \cdots + 2^{n-1}$$

\- ## Proof

Base case – $n = 1$

$2^1 - 1 = 2 - 1 = 1 \Rightarrow$ holds

Assume $2^k - 1 = 1 + 2 + 2^2 + \cdots + 2^{k-1}$

consider $k+1$

$$2^{k+1} = 2 \cdot 2^k$$
$$= 2(2^k - 1) + 2$$

$= 2(1 + 2 + \cdots + 2^{k-1}) + 2$ (by IH)

$= 2 + 2^2 + \cdots + 2^k + 2$

$\Rightarrow 2^{k+1} - 1 = 2 + 2^2 + \cdots + 2^k + 2 - 1$

$= 2 + 2^2 + \cdots + 2^k + 1$

$= 1 + 2 + 2^2 + \cdots + 2^{(k+1)-1}$

$\Rightarrow$ holds $\forall k$ by induction