Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ $\quad T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ 2y \end{pmatrix}$

let $\beta = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ be the basis for the range and domain

$$T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 0\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} = 1\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 2\begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \Rightarrow \quad Mat(T) = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ 2y \end{pmatrix}$$

let $\beta' = \left\{ \begin{pmatrix} 5 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$ be a basis for the domain of T

$\beta = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ be a basis for the range of T

(so $T: \beta' \to \beta$, write $\begin{pmatrix} a \\ b \end{pmatrix}_{\beta'} = a\begin{pmatrix} 5 \\ 3 \end{pmatrix} + b\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\begin{pmatrix} a \\ b \end{pmatrix} = a\begin{pmatrix} 1 \\ 0 \end{pmatrix} + b\begin{pmatrix} 0 \\ 1 \end{pmatrix}$)

$$T\begin{pmatrix} 5 \\ 3 \end{pmatrix} = \begin{pmatrix} 8 \\ 6 \end{pmatrix} = 8\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 6\begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \Rightarrow \quad Mat(T)' = \begin{pmatrix} 8 & 1 \\ 6 & 0 \end{pmatrix}$$

$$T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 0\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

How do we rewrite $\begin{pmatrix} x \\ y \end{pmatrix}$ in our new basis β'

$$\begin{pmatrix} x \\ y \end{pmatrix} = \tfrac{1}{3}y\begin{pmatrix} 5 \\ 3 \end{pmatrix} + (x - \tfrac{5}{3}y)\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \tfrac{1}{3}y \\ x - \tfrac{5}{3}y \end{pmatrix}_{\beta'}$$

$$\begin{pmatrix} 8 & 1 \\ 6 & 0 \end{pmatrix}\begin{pmatrix} \tfrac{1}{3}y \\ x - \tfrac{5}{3}y \end{pmatrix}_{\beta'} = \begin{pmatrix} \tfrac{8}{3}y + x - \tfrac{5}{3}y \\ \tfrac{6}{3}y \end{pmatrix} = \underbrace{\begin{pmatrix} x+y \\ 2y \end{pmatrix}}_{\text{the rule for } T}$$

$T: \beta' \to \beta$

$\Rightarrow Mat(T)' \times$ something in β' = something in β

let $\beta' = \left\{ \begin{pmatrix} 5 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$ be a basis of the domain of T

let $\beta'' = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \end{pmatrix} \right\}$ be a basis of the range of T

(write $\begin{pmatrix} a \\ b \end{pmatrix}_{\beta'} = a\begin{pmatrix} 5 \\ 3 \end{pmatrix} + b\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\quad \begin{pmatrix} a \\ b \end{pmatrix}_{\beta''} = a\begin{pmatrix} 1 \\ 1 \end{pmatrix} + b\begin{pmatrix} 0 \\ 2 \end{pmatrix}$)

$$T\begin{pmatrix} 5 \\ 3 \end{pmatrix} = \begin{pmatrix} 8 \\ 6 \end{pmatrix} = 8\begin{pmatrix} 1 \\ 1 \end{pmatrix} - 1\begin{pmatrix} 0 \\ 2 \end{pmatrix} \quad \Rightarrow \quad \mathrm{Mat}(T)'' = \begin{pmatrix} 8 & 1 \\ -1 & -\frac{1}{2} \end{pmatrix}$$

$$T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1\begin{pmatrix} 1 \\ 1 \end{pmatrix} - \tfrac{1}{2}\begin{pmatrix} 0 \\ 2 \end{pmatrix}$$

① consider $x\begin{pmatrix} 1 \\ 0 \end{pmatrix} + y\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ as in standard basis

$$x\begin{pmatrix} 1 \\ 0 \end{pmatrix} + y\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \tfrac{1}{3}y\begin{pmatrix} 5 \\ 3 \end{pmatrix} + (x - \tfrac{5}{3}y)\begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{3}y \\ x - \frac{5}{3}y \end{pmatrix}_{\beta'}$$

$$\begin{pmatrix} 8 & 1 \\ -1 & -\frac{1}{2} \end{pmatrix}\begin{pmatrix} \frac{1}{3}y \\ x - \frac{5}{3}y \end{pmatrix}_{\beta'} = \begin{pmatrix} \frac{8}{3}y + x - \frac{5}{3}y \\ -\frac{1}{3}y - \frac{1}{2}x + \frac{5}{6}y \end{pmatrix}_{\beta''}$$

← $T: \beta' \to \beta''$
$\mathrm{Mat}(T)'' \times$ something in β'
$=$ something in β''

$$= \begin{pmatrix} x + y \\ -\frac{1}{2}x + \frac{1}{2}y \end{pmatrix}_{\beta''}$$

rewrite to standard coords

$$= (x+y)\begin{pmatrix} 1 \\ 1 \end{pmatrix} + (-\tfrac{1}{2}x + \tfrac{1}{2}y)\begin{pmatrix} 0 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} x + y + 0 \\ x + y - x + y \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} x + y \\ 2y \end{pmatrix}}_{\text{the rule for } T}$$