Inherited Properties

Groups

X closure - $\forall x,y \in G \quad x*y \in G$

✓ associativity - $\forall x,y,z \in G \quad (x*y)*z = x*(y*z)$

X identity - $\exists 1 \in G$ st $1*x = x*1 \quad \forall x \in G$

X inverses - $\forall x \in G \quad \exists x^{-1} \in G$ st $x*x^{-1} = x^{-1}*x = 1$

Rings + Fields

X closure - ~~$\forall a,b \in G$~~ $\forall a,b \in R$
$a+b \in R$ and $ab \in R$

✓ A1 - $\forall a,b \in R \quad a+b = b+a$

✓ A2 - $\forall a,b,c \in R \quad (a+b)+c = a+(b+c)$

X A3 - $\exists 0 \in R$ st $a+0 = 0+a = a \quad \forall a \in R$

X A4 - $\forall a \in R \quad \exists -a \in R$ st $a+(-a) = (-a)+a = 0$

✓ M2 - $\forall a,b,c \in R \quad (ab)c = a(bc)$

✓ D - $\forall a,b,c \in R \quad a(b+c) = ab+ac$
$(a+b)c = ac+bc$

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X M3 - $\exists 1 \in F \quad 1 \neq 0$ st $1a = a1 = a \quad \forall a \in F$

X M4 - $\forall a \in F \quad a \neq 0 \quad \exists a^{-1} \in F$ st $aa^{-1} = a^{-1}a = 1$

$\exists$ = there exists
$\forall$ = for all

Proofs and Counter Examples

In each case we give a group/field/ring and a subset S

Groups

X closure/binary op

maybe check this is a group for some practice

$G = M_{2\times 2}(\mathbb{R})$ op = usual matrix addition

$S = \{A \in G \mid \det(A) = 1\} \subseteq G$

$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \in S$

$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & \frac{3}{2} \end{pmatrix} \notin S$

✓ associativity

let G be a group, $H \subseteq G$ which has the same binary op (H closed under the op)

let $a, b, c \in H$

$\Rightarrow a, b, c \in G$ as $H \subseteq G$

$\Rightarrow \underbrace{(ab)c = a(bc)}_{\text{in } G}$ since G a group so associative

since the op of H and G are the same $(ab)c$ and $a(bc)$ are the same in H as in G

$\Rightarrow \underbrace{(ab)c = a(bc)}_{\text{in } H}$

$\Rightarrow H$ is associative

X identity

$G = M_{2\times 2}(\mathbb{R})$ op = usual matrix addition

$id = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ ← practice check

$S = \{ A \in G \mid \det A = 1 \}$

assume $\begin{pmatrix} w & x \\ y & z \end{pmatrix}$ is the id

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} w & x \\ y & z \end{pmatrix} = \begin{pmatrix} a+w & b+x \\ c+y & d+z \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$\Rightarrow w = x = y = z = 0$

$\Rightarrow \begin{pmatrix} w & x \\ y & z \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \notin S$

X inverse

* $G = M_{2\times 2}(\mathbb{R})$ op = usual addition

$-\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix}$

$S = \{ A \in G \mid \det A = 1 \}$ (as above)

S has no identity $\Rightarrow$ no inverses

* $G = \mathbb{R}$ op = multiplication

id = 1 ← we need to know the id before we can think of inverses

$a \in \mathbb{R} \setminus \{0\}$ $a^{-1} = 1/a \in \mathbb{R}$

$\mathbb{Z} \subseteq \mathbb{R}$

id = 1 as before

$a \in \mathbb{Z}$ $a \neq 1, -1, 0$

Suppose $x = a^{-1}$

$\Rightarrow ax = 1$

$\Rightarrow |ax| = |a||x| = 1$

$\Rightarrow |x| = \frac{1}{|a|} < 1$ and $\neq 0$

\# because

if $x \in \mathbb{Z} \Rightarrow |x| = 0$ or $|x| \geq 1$

either proof works

$2 \in \mathbb{Z}$

$2 \cdot x = 1$

$\Rightarrow x = \frac{1}{2} \notin \mathbb{Z}$

Rings + Fields

X closure

$S_1, S_2 \subseteq M_{2\times2}(\mathbb{R})$ usual + and x of matrices

$S_1 = \{ A \in G \mid \det(A) = 1 \}$

$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \in S_1$

$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & \frac{3}{2} \end{pmatrix} \notin S_1$

$S_2 = \left\{ \begin{pmatrix} \lambda & 2\lambda \\ 0 & \lambda \end{pmatrix} \mid \lambda \in \mathbb{R} \right\}$

$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 4 \\ 0 & 2 \end{pmatrix} \in S_2$

$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 4 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 8 \\ 0 & 2 \end{pmatrix} \notin S_2$

A1, A2, M2, D

These are similar to the proof of associativity for groups. Try writing your own and email them to me if you want them checked ü

X A3

invertible matrices

$S = \{ A \in M_{2\times2}(\mathbb{R}) \mid \det(A) \neq 0 \} \subseteq M_{2\times2}(\mathbb{R})$

$\underline{0}$ in $M_{2\times2}(\mathbb{R})$ is $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ usual + and x of matrices

assume $\underline{0} = \begin{pmatrix} w & x \\ y & z \end{pmatrix} \in S$ and $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in S$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} w & x \\ y & z \end{pmatrix} = \begin{pmatrix} a+w & b+x \\ c+y & d+z \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$\Leftrightarrow \quad w = x = y = z = 0$

$\Leftrightarrow \quad \begin{pmatrix} w & x \\ y & z \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

but $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \notin S$

so no additive identity

X A4

$M_{2\times 2}(\mathbb{R})$ usual addition and multi in $\mathbb{R}$

$\underline{0} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ (as in A3)

$\Rightarrow -\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix}$

so holds in $M_{2\times 2}(\mathbb{R})$

$S = \left\{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \middle| \; a \geq 0 \right\} \subseteq M_{2\times 2}(\mathbb{R})$

$\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in S$

$\Rightarrow \underline{0} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

remember we need to find the id before we can think of inverses

assume $\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \in S$ and let $\begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix}$ be the inverse

$\Rightarrow \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a+b & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$\Rightarrow a + b = 0$

$\Rightarrow b = -a$

when $a \neq 0$ $\quad b = -a < 0$

$\Rightarrow \begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} -a & 0 \\ 0 & 0 \end{pmatrix} \notin S$

X M3

$M_{2\times2}(\mathbb{R})$ usual $+$ and $\times$

$\underline{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\leftarrow$ easy to check

$S = \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & 2\lambda \end{pmatrix} \middle| \lambda \in \mathbb{R} \right\} \subseteq M_{2\times2}(\mathbb{R})$

(S is a ring (but not a field) - why not check this for some practice)

taking $\lambda = 1$ gives $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \in S$

suppose $\exists \underline{1} \in S$ $\quad \underline{1} = \begin{pmatrix} x & 0 \\ 0 & 2x \end{pmatrix}$

$\Rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} x & 0 \\ 0 & 2x \end{pmatrix} = \begin{pmatrix} x & 0 \\ 0 & 4x \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$

$\Rightarrow x = 1 \qquad 4x = 2$

$\Downarrow$

$x = 1/2$

$\# \quad 1 \neq 1/2$

$\Rightarrow$ S has no id under multiplication

X M4

$\mathbb{R}$ usual $+$ and $\times$

multi id $= 1$

$a \in \mathbb{R} \quad a \neq 0 \qquad a^{-1} = 1/a$

$\mathbb{Z} \subseteq \mathbb{R}$

$x \cdot 1 = 1 = 1 \cdot x \qquad \forall x \in \mathbb{Z}$

$\Rightarrow$ multi id $= 1$

now we have the id we can check inverses

$2 \in \mathbb{Z}$ suppose $2^{-1} = x$

$2 \cdot x = 1$

$\Rightarrow x = 1/2 \notin \mathbb{Z}$

$\Rightarrow \mathbb{Z}$ doesn't have multiplicative inverses