$$A = \begin{pmatrix} 1 & 2 & 1 \\ 6 & -1 & 0 \\ -1 & -2 & -1 \end{pmatrix}$$

$$\det(A - \lambda I) = \det \begin{pmatrix} 1-\lambda & 2 & 1 \\ 6 & -1-\lambda & 0 \\ -1 & -2 & -1-\lambda \end{pmatrix}$$

expanding down the 3rd column

$$= 1\left(6(-2) - -1(-1-\lambda)\right) - 0 + (-1-\lambda)\left((1-\lambda)(-1-\lambda) - 2(6)\right)$$

$$= -12 + (-1-\lambda) + (-1-\lambda)(-1+\lambda-\lambda+\lambda^2 - 12)$$

$$= -12 - 1 - \lambda - (1+\lambda)(-13+\lambda^2)$$

$$= -13 - \lambda + 13 - \lambda^2 + 13\lambda - \lambda^3$$

$$= 12\lambda - \lambda^2 - \lambda^3$$

$$= \lambda(12 - \lambda - \lambda^2)$$

$$= -\lambda(\lambda^2 + \lambda - 12)$$

$$= -\lambda(\lambda+4)(\lambda-3)$$

$\Rightarrow$ Evals $\lambda = 0 \quad \lambda = -4 \quad \lambda = 3$

$\underline{\lambda = 0}$

~~$A\lambda$~~ $Av = \lambda v$

$\Rightarrow Av = \underline{0}$

let $v = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 1 & 2 & 1 \\ 6 & -1 & 0 \\ -1 & -2 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \left.\begin{array}{ll} x + 2y + z = 0 & \text{— } ① \\ 6x - y = 0 & \text{— } ② \\ -x - 2y - z = 0 & \text{— } ③ \end{array}\right\}$$

can solve using simultaneous eqns or by row ops

notice $① = -③$

$x + 2y + z = 0$ — ① $\qquad 6x = y$ — ②

$\Rightarrow x + 2(6x) + z = 0$

$\Rightarrow 13x + z = 0$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ 6x \\ -13x \end{pmatrix} = x\begin{pmatrix} 1 \\ 6 \\ -13 \end{pmatrix}$$

$\Rightarrow$ ~~Soln~~ ~~Evec~~ For $\lambda = 0$ $\quad \left\{ \begin{pmatrix} \lambda \\ 6\lambda \\ -13\lambda \end{pmatrix} \middle| \ \lambda \in \mathbb{R} \right\}$

$\lambda = -4$

$Av = \lambda v$

$\Rightarrow Av = -4v$

$\Rightarrow (A + 4I)v = \underline{0}$

$$\Rightarrow \begin{pmatrix} 1+4 & 2 & 1 \\ 6 & -1+4 & 0 \\ -1 & -2 & -1+4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \underbrace{\begin{pmatrix} 5 & 2 & 1 \\ 6 & 3 & 0 \\ -1 & -2 & 3 \end{pmatrix}}_{①} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \underbrace{\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}}_{②}$$

Let's try solving this one with row ops

$$\left(\begin{array}{ccc|c} 5 & 2 & 1 & 0 \\ 6 & 3 & 0 & 0 \\ -1 & -2 & 3 & 0 \end{array}\right)$$

$$\xrightarrow{r_3 \leftrightarrow r_1} \left(\begin{array}{ccc|c} -1 & -2 & 3 & 0 \\ 6 & 3 & 0 & 0 \\ 5 & 2 & 1 & 0 \end{array}\right)$$

$$\xrightarrow{r_1 \to -r_1} \left(\begin{array}{ccc|c} 1 & 2 & -3 & 0 \\ 6 & 3 & 0 & 0 \\ 5 & 2 & 1 & 0 \end{array}\right)$$

$$\xrightarrow{r_2 \to r_2 - 6r_1} \left(\begin{array}{ccc|c} 1 & 2 & -3 & 0 \\ 0 & -9 & 18 & 0 \\ 5 & 2 & 1 & 0 \end{array}\right)$$

$$\xrightarrow{r_3 \to r_3 - 5r_1} \left(\begin{array}{ccc|c} 1 & 2 & -3 & 0 \\ 0 & -9 & 18 & 0 \\ 0 & -8 & 16 & 0 \end{array}\right)$$

$$\xrightarrow{r_2 \to -\frac{1}{9} r_2} \left(\begin{array}{ccc|c} 1 & 2 & -3 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & -8 & 16 & 0 \end{array}\right)$$

$$\xrightarrow{r_3 \to r_3 + 8r_2} \left(\begin{array}{ccc|c} 1 & 2 & -3 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{r_1 \to r_1 - 2r_2} \left(\underbrace{\begin{array}{ccc} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{array}}_{B}\right|\left.\begin{array}{c} 0 \\ 0 \\ 0 \end{array}\right)$$

Hence solving

$\cancel{AV=0}$

$(A + 4I)\,V = \underline{0}$ is the same as $BV = \underline{0}$

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

let $V = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$\Rightarrow \quad x + z = 0 \qquad y - 2z = 0$

$\Rightarrow \quad \cancel{x}\; z = -x \qquad y = 2z = 2(-x) = -2x$

$$\Rightarrow \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ -2x \\ -x \end{pmatrix} = x\begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}$$

$\Rightarrow$ for $\lambda = -4$ E vecs $= \left\{ \begin{pmatrix} \lambda \\ -2\lambda \\ -\lambda \end{pmatrix} \,\middle|\, \lambda \in \mathbb{R} \right\}$

Check

$$AV = \begin{pmatrix} 1 & 2 & 1 \\ 6 & -1 & 0 \\ -1 & -2 & -1 \end{pmatrix}\begin{pmatrix} \lambda \\ -2\lambda \\ -\lambda \end{pmatrix} = \begin{pmatrix} \lambda + 2(-2\lambda) + 1(-\lambda) \\ 6(\lambda) - 1(-2\lambda) + 0 \\ -1(\lambda) - 2(-2\lambda) - 1(-\lambda) \end{pmatrix} = \begin{pmatrix} -4\lambda \\ 8\lambda \\ 4\lambda \end{pmatrix}$$

$$= -4\begin{pmatrix} \lambda \\ -2\lambda \\ -\lambda \end{pmatrix}$$

* Finding Evals

For hard to factorise deg 3 polys like

$$p(\lambda) = \lambda^3 - 12\lambda - 16 = 0$$

remember that roots are divisois of -16.

As $\deg(p(\lambda)) = 3$ there must be one real root (because odd roots come in pairs).

So try $\{\pm 1, \pm 2, \pm 4, \pm 8, \pm 16\}$

$\lambda = 1 \quad p(1) = 1 - 12 - 16 \neq 0$

$\lambda = -1 \quad p(-1) = -1 + 12 - 16 \neq 0$

$\vdots \qquad \vdots$

$\lambda = 4 \quad p(4) = 4^3 - 12(4) - 16 = 0$

$\Rightarrow \lambda = 4$ is a root

$$\left.\begin{aligned}
\lambda^3 - 12\lambda - 16 &= (\lambda - 4)(\lambda^2) + 4\lambda^2 - 12\lambda - 16 \\
&= (\lambda - 4)(\lambda^2 + 4\lambda) + 16\lambda - 12\lambda - 16 \\
&= (\lambda - 4)(\lambda^2 + 4\lambda) + 4\lambda - 16 \\
&= (\lambda - 4)(\lambda^2 + 4\lambda + 4)
\end{aligned}\right\}$$

Let me know if this is tricky and we can go over it

need roots of $\lambda^2 + 4\lambda + 4$

$$\lambda = \frac{-4 \pm \sqrt{4^2 - 4(1)(4)}}{2(1)} = \frac{-4 \pm 0}{2} = \underbrace{-2}_{\text{a repeated root}}$$

$\Rightarrow$ roots $= \lambda = 4, \ \lambda = -2, \ \lambda = -2$